THE HIERARCHY OF ASYMPTOTIC DOMINANCE

ADRIAN PĂCURAR

Contents

1. Introduction

Often times in calculus students are tasked with evaluating limits of the form

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)}$$

where the numerator and denominator are polynomials $P(x)=p_nx^n+p_{n-1}x^{n-1}+\cdots+p_1x+p_0$ and $Q(x)=q_mx^m+q_{m-1}x^{m-1}+\cdots+q_1x+q_0$, so that $\deg P=n$ and $\deg Q=m$. After computing several limits by hand, one usually notices the shortcut:

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)}=\begin{cases}0 & \deg P<\deg Q\\ p_n/q_n & \deg P=\deg Q=n\\ \pm\infty & \deg P>\deg Q\end{cases}$$

The intuition behind this is that when $\deg P<\deg Q$ is that the leading term of Q *dominates* the leading term of P, so the infinity in the denominator is of a higher "order", bringing the entire fraction to zero. The opposite happens when $\deg P>\deg Q$, and when the degrees are equal, the orders of infinity in the numerator and denominator are the same, so ignoring all the lower order terms, one can cancel the leading terms, which yields the ratio of the leading coefficients of P and Q. While this isn't very rigorous at all, it is later proven to work using L'Hôpital's rule.

A similar shortcut can be used if the numerator and denominator are no longer polynomials, but more complicated functions involving logarithms, exponentials, roots, etc. The intuition is similar, and it has to do with the asymptotic dominance of the numerator versus denominator.

Theorem 1. (Asymptotic Dominance) Suppose that $b>1$, $p>0$, and $a>1$ are real numbers. Then, as $n\to\infty$, we have

$$\log_b(n)<n^p<a^n<n!<n^n \tag{1}$$

where n runs over the positive integers. The $<$ sign is to be interpreted in the limiting sense:

$$\lim\frac{\log n}{n^p}=\lim\frac{n^p}{a^n}=\cdots=0 \qquad \text{and} \qquad \lim\frac{n^p}{\log n}=\lim\frac{a^n}{n^p}=\cdots=\infty$$

Note: the reason we are restricting n solely to integers is because of the factorial. We would require the Gamma function in order to expand it to real numbers, which is more complicated to study.

2. Log versus Polynomial

To prove the polynomial dominance over the logarithm, we are looking at the limit

$$\lim_{x\to\infty}\frac{\log_b(x)}{x^p}$$

where $b>1$ (otherwise the logarithm would be a decreasign function) and $p>0$ (otherwise the polynomial would be decreasing of the type $1/x^p$). To compute the above limit, notice that under the conditions mentioned, both the numerator and denominator tend to infinity, so we can freely apply L'Hôpital's rule to get

$$\lim_{x\to\infty}\frac{\log_b(x)}{x^p}=\lim_{x\to\infty}\frac{\dfrac{1}{x\ln b}}{px^{p-1}}=\lim_{x\to\infty}\frac{1}{px\cdot x^{p-1}\ln b}=\lim_{x\to\infty}\frac{1}{px^p\ln b}=0$$

thus proving the asymptotic dominance of x^p over $\log x$.

Some applications of this are the following limit computations:

$$\lim\frac{\sqrt{x}}{\ln x+x+1}=0\qquad\lim\frac{\sqrt[3]{x}+1}{\ln x+8}=\infty\qquad\lim\frac{\sqrt{2x^2+3x+1}}{1+3x+5\ln x}=\frac{\sqrt{2}}{3}$$

We saw that $\log x<x$ in the asymptotic sense. But what if we try to compare $(\log x)^n$ to x instead? Suppose n is a positive integer, and consider the limit (using L'Hôpital's rule):

$$\lim_{x\to\infty}\frac{(\log x)^n}{x}=\lim_{x\to\infty}\frac{n(\log x)^{n-1}\cdot\dfrac{1}{x}}{1}=\lim_{x\to\infty}\frac{n(\log x)^{n-1}}{x}=\cdots=\lim_{x\to\infty}\frac{n!}{x}=0$$

and a similar computation will show that $(\log x)^n<x^p$ for any $p>0$. This means that even if we raise the logarithm to some power, it will eventually lose against the polynomial functions.

It is worth mentioning that we can actually insert another family of functions between $\ln x$ and x in the dominance sequence. For $q\in(0,1)$, we have

$$\ln x<x^{1-q}<\frac{x}{\ln x}<x$$

so the function $\frac{x}{\ln x}$ tends to infinity more slowly than any power of x (or than x^p for $p>1$), but faster than x^{1-q}, i.e. than any power of x less than the first. The function

$$\frac{x}{\ln x}$$

appears in many area of mathematics, especially in the study of prime numbers.

3. Polynomial versus Exponential

To show the exponential dominance over the polynomial (and thus over the logarithm, we use L'Hôpital's rule to compute the limit

$$\lim_{x\to\infty} \frac{x^p}{a^x}$$

Note that we will need to apply L'Hôpital's several times, until the exponent of x becomes less than zero (or, if p is an integer, until the exponent of p reduces to zero). In either case, we have a computation of the form

$$\lim_{x\to\infty} \frac{x^p}{a^x} = \lim_{x\to\infty} \frac{px^{p-1}}{a^x \ln a} = \lim_{x\to\infty} \frac{p(p-1)x^{p-2}}{a^x(\ln a)^2} = \begin{cases} \lim_{x\to\infty} \dfrac{p!}{a^x(\ln a)^p} & \text{if } p \in \mathbb{Z} \\ \lim_{x\to\infty} \dfrac{p(p-1)\dots(p-N+1)x^{p-N}}{a^x(\ln a)^N} & \text{if } p \notin \mathbb{Z} \end{cases}$$

It is easy to see that whenever p is an integer, we end up with a constant ($p!$) divided by an exponential which tends to infinity, so the limit will be zero. If p is not an integer, after repeating L'Hôpital's N times (where N is the first integer so that $p-N$ is a negative exponent), the function x^{p-N} will go to zero (belongs to the family of curves $1/x^n$, which approach zero as x goes to infinity), so the limit is again zero. Hence

$$\lim_{x\to\infty} \frac{x^p}{a^x} = 0$$

so the exponential function a^x dominates the polynomial x^p.

Infinite series can also be used to illustrate the asymptotic dominance of exponential functions over polynomials. Consider the functions e^x and x^n, and recall the series expansion of e^x:

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \dots$$

so that e^x/x^n is going to satisfy (if we omit all other positive terms)

$$e^x x^{-n} > \frac{x}{(n+1)!}$$

when x is positive. Now, letting $x \to \infty$, we see that $e^x x^{-n} \to \infty$, as expected.

Since e^x tends to infinity more rapidly than any power of x, this implies that $\ln x$ tends to infinity more slowly than any power of x, as this is the inverse function of e^x (how are their graphs related?).

4. Exponential versus Factorial

Since we would like to avoid the use of the Gamma function (a generalization of factorials to real and complex numbers, not just integers), we are looking at a limit of the type

$$\lim_{n\to\infty} \frac{n!}{a^n}$$

as n runs over the integers. This means we can no longer compute such a limit using L'Hôpital's rule, as this would require continuity in the limiting variable (this is why we can take derivative with L'Hôpital's). Our goal is to show that this limit equals infinity.

To give some intuition (though not a complete proof), consider the case where $a = 2$:

$$L = \lim_{n\to\infty} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{1\times 2\times 3\times 4\times\cdots\times n}{2\times 2\times 2\times 2\times\cdots\times 2} = \lim_{n\to\infty} \frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}\cdot\frac{4}{2}\cdots\frac{n-1}{2}\cdot\frac{n}{2}$$

so as n gets very large, it is very easy to see that this limit will be ∞.

A more rigorous way to prove that $\lim_{n\to\infty} \frac{n!}{a^n} = \infty$ is to consider the sequence

$$b_n = \frac{n!}{a^n}$$

What happens if we take ratios of consecutive terms?

$$\frac{b_{n+1}}{b_n} = \frac{\frac{(n+1)!}{a^{n+1}}}{\frac{n!}{a^n}} = \frac{(n+1)!}{a^{n+1}}\cdot\frac{a^n}{n!} = \frac{n+1}{a}$$

and so

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{n+1}{a} = \infty$$

How do we interpret this result? As n gets very large, the above computation is basically saying that to get from b_n to b_{n+1}, you have to multiply b_n by an increasingly VERY large number (∞, as the limit shows). Hence

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n!}{a^n} = \infty$$

so the factorial is asymptotically dominant over the exponential.

5. Factorial versus n^n

To establish the asymptotic dominance of n^n over $n!$, we consider the limit

$$\lim_{n\to\infty} \frac{n!}{n^n}$$

and expanding the numerator and denominator as we did before, we get

$$\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{1\times 2\times 3\times 4\times\cdots\times n}{n\times n\times n\times n\times\cdots\times n}$$

so intuition tells us that this limit is zero. To introduce some rigor to our argument, let

$$A_n = \frac{n!}{n^n}$$

Now we borrow an idea from the root test in infinite series: if we can show that the limit of the n-th root of A_n is zero, then it must be the case that A_n goes to zero, otherwise the n-th root would approach one. To understand this argument, simply plot the function $x^{1/n}$ on the interval $[0, 2]$ for $n = 10, 100, 1000, 10000$, and notice that the graphs get closer and closer to the horizontal line $y = 1$ as n increases. In other words, we are making use of the fact that

$$\lim_{y\to\infty} \alpha^{1/y} = \begin{cases} 0 & \alpha = 0 \\ 1 & \alpha > 0 \end{cases}$$

which we can easily prove using L'Hôpital's rule. So our goal is to show that the n-th root of A_n is zero. Consider the limit

$$L = \lim_{n\to\infty} (A_n)^{1/n} = \lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{(n!)^{1/n}}{n}$$

Taking natural log and using the rules of logarithms to break up the fraction gives

$$\ln L = \lim_{n\to\infty} \ln\left(\frac{(n!)^{1/n}}{n}\right) = \lim_{n\to\infty} \left(\frac{1}{n}\ln(n!) - \ln n\right)$$

The factorial is just a product, so this breaks up even further as

$$\begin{aligned} \ln L &= \lim_{n\to\infty} \left(\frac{\ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n)}{n} - \ln n\right) \\ &= \lim_{n\to\infty} \left(\frac{\ln 1}{n} + \cdots + \frac{\ln(n-1)}{n} + \frac{\ln n}{n} - \ln n\right) \end{aligned}$$

and now we can use the asymptotic dominance of polynomials over logarithms to argue that every fraction goes to zero, while the remaining term $-\ln n$ goes to minus infinity. Hence

$$\ln L = -\infty$$

which implies that $L = 0$, as desired.

An alternate proof, which may be more clear and easier to understand, is to instead consider the sequence

$$B_n = \frac{n^n}{n!}$$

and take the limit of the ratio of consecutive terms B_{n+1}/B_n (as we did when we compared a^n to $n!$ in the previous section). The idea here is to try and understand by what factor does B_{n+1} differ from B_n. We have:

$$\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n(n+1)}{(n+1)\cdot n!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

and computing this is very easy, since it is equal to

$$\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

so for n very large, B_{n+1} is approximately e times B_n - an exponential growth! Hence

$$\lim \frac{n^n}{n!} = \infty$$

as expected.