# THE AVERAGE VALUE OF A SEQUENCE 

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## 1. Introduction

When calculus students study power series of the form

$$
f(x)=\sum_{n=0}^{\infty} a_{n}(x-c)^{n}
$$

they learn to find the radius of convergence $R$ by using either the ratio or the root test. The computation boils down to simply looking at the coefficients. When the $a_{n}$ are positive (otherwise take the absolute value of the terms), we have

$$
\lim _{n \rightarrow \infty} \sqrt[n]{a_{n}}=\frac{1}{R}=\lim _{n \rightarrow \infty} \frac{a_{n+1}}{a_{n}}
$$

Most calculus books mention this shortcut, but very few (I haven't seen one yet) provide a proof of this fact. So let's take the following theorem as our starting point:

Theorem 1. Suppose $a_{n}$ is a sequence of positive terms, and that

$$
\begin{equation*}
L=\lim _{n \rightarrow \infty} \frac{a_{n+1}}{a_{n}} \tag{1}
\end{equation*}
$$

exists. Then $\lim _{n \rightarrow \infty} \sqrt[n]{a_{n}}$ exists and is also equal to $L$, i.e

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sqrt[n]{a_{n}}=L \tag{2}
\end{equation*}
$$

Before providing a proof of this, we will need an additional lemma.
Lemma 1. Suppose $r_{n}$ is a sequence of positive terms converging to $\lim _{n \rightarrow \infty} r_{n}=R$. Then

$$
\lim _{n \rightarrow \infty} \sqrt[n]{r_{1} \cdot r_{2} \ldots r_{n}}=R
$$

The motivation behind this lemma was equation (2): I wanted to somehow introduce a product under the root, where all the terms would cancel except $a_{n}$. In other words, if we define a new sequence $r_{n}=a_{n} / a_{n-1}$, notice that $\lim r_{n}=L$, so by Lemma 1 we obtain

$$
L=\lim _{n \rightarrow \infty} \sqrt[n]{r_{1} \cdot r_{2} \ldots r_{n}}=\lim _{n \rightarrow \infty} \sqrt[n]{\frac{a_{1}}{a_{0}} \cdot \frac{a_{2}}{a_{1}} \ldots \frac{a_{n}}{a_{n-1}}}=\lim _{n \rightarrow \infty} \sqrt[n]{\frac{a_{n}}{a_{0}}}=\lim _{n \rightarrow \infty} \sqrt[n]{a_{n}}
$$

thus proving Theorem 1. Of course, we still need to argue that the lemma is true.

## 2. The Average Value Of A Sequence

One can prove Lemma 1 by starting with a positive sequence $r_{n} \rightarrow R$, and considering the limit

$$
R^{\prime}=\lim _{n \rightarrow \infty} \sqrt[n]{r_{1} \cdot r_{2} \ldots r_{n}}
$$

which, after applying natural log, becomes

$$
\begin{equation*}
\log R^{\prime}=\lim _{n \rightarrow \infty} \frac{1}{n}\left(\log r_{1}+\log r_{2}+\cdots+\log r_{n}\right) \tag{3}
\end{equation*}
$$

The expression inside the limit should ring a bell: it looks a lot like the average value of the first $n$ terms of the sequence $\log r_{n}$. If $r_{n} \rightarrow R$, applying $\log$ gives us $\log r_{n} \rightarrow \log R$. What can be said about the average value of the first $n$ terms of $\log r_{n}$ ? Intuition tells us that for large $n$, the majority of the terms have to be close to the limit, so it is reasonable to expect that, on average, the sequence has the value of its limit. This means that the limit in (3) evaluates to $\log R$, thus proving Lemma 1.

While the intuitive argument gives the correct result, how do we prove such a thing more rigorously? First, we need to understand what one means by the limit of a sequence.

Definition 1. Let $\left\{a_{n}\right\}$ be a sequence of real numbers. Then $\lim _{n \rightarrow \infty} a_{n}=L$ if and only if for every $\epsilon>0$, there exists some $N_{\epsilon}$ such that if $k \geq N_{\epsilon}$, then

$$
\left|a_{k}-L\right|<\epsilon
$$

To understand the definition, one should think of $\epsilon$ as the error (or difference) between the tail terms of $a_{n}$ and the limit $L$. Then if the sequence converges to $L$, the definition is simply saying all the terms are within $\epsilon$ of the limit. In more precise terms, all the terms with index higher than some $N_{\epsilon}$ are going to be within $\epsilon$ of the limit (so the error is at most $\epsilon)$. This is what is meant by writing $\left|a_{k}-L\right|<\epsilon$.

Armed with this definition, we are ready to try and prove that our intuitive understanding regarding the average value of a sequence is actually correct.

Theorem 2. The average value of a convergent sequence is the limit of the sequence.

Proof. Suppose $\lim a_{n}=L$, and fix $\epsilon>0$. By the definition, we can find $N_{\epsilon}$ such that

$$
k \geq N_{\epsilon} \quad \Longrightarrow \quad\left|a_{k}-L\right|<\epsilon
$$

If we consider the sequence

$$
b_{n}=\frac{a_{1}+a_{2}+\cdots+a_{n}}{n}
$$

so that $b_{n}$ represents the average of the first $n$ terms of our original sequence $a_{n}$, then the proof of our theorem reduces to showing that $b_{n} \rightarrow L$. We would like to make use of our
definition, so we begin by analyzing the error term $\left|b_{n}-L\right|$.

$$
\begin{aligned}
\left|b_{n}-L\right| & =\left|\frac{a_{1}+a_{2}+\cdots+a_{n}}{n}-L\right| \\
& =\left|\frac{a_{1}+a_{2}+\cdots+a_{n}}{n}-\frac{n L}{n}\right| \quad \text { (common denominator) } \\
& =\left|\frac{a_{1}+a_{2}+\cdots+a_{n}-n L}{n}\right| \quad \text { (combine fractions) } \\
& =\left|\frac{\left(a_{1}-L\right)+\left(a_{2}-L\right)+\cdots+\left(a_{n}-L\right)}{n}\right| \quad \text { (grouping) }
\end{aligned}
$$

The grouping is made possible since we have precisely $n$ copies of $a_{i}$, and $n$ copies of $L$. Now, we can apply the triangle inequality to obtain (assuming $n$ is large enough to surpass $N_{\epsilon}$ ):

$$
\begin{aligned}
\left|b_{n}-L\right| & \leq\left|\frac{a_{1}-L}{n}\right|+\left|\frac{a_{2}-L}{n}\right|+\cdots+\left|\frac{a_{N_{\epsilon}-1}-L}{n}\right|+\left|\frac{a_{N_{\epsilon}}-L}{n}\right|+\cdots+\left|\frac{a_{n}-L}{n}\right| \\
& =\sum_{k=1}^{N_{\epsilon}-1}\left|\frac{a_{k}-L}{n}\right|+\sum_{k=N_{\epsilon}}^{n}\left|\frac{a_{k}-L}{n}\right| \\
& <\sum_{k=1}^{N_{\epsilon}-1} \frac{\left|a_{k}-L\right|}{n}+\sum_{k=N_{\epsilon}}^{n} \frac{\epsilon}{n} \quad\left(\text { since }\left|a_{k}-L\right|<\epsilon \text { for } k \geq N_{\epsilon}\right)
\end{aligned}
$$

In other words, we find that

$$
\left|b_{n}-L\right|<\frac{1}{n} \sum_{k=1}^{N_{\epsilon}-1}\left|a_{k}-L\right|+\frac{n-N_{\epsilon}+1}{n} \epsilon
$$

Now, notice that the sum of the errors for the beginning terms of the sequence (first $N_{\epsilon}-1$ terms) is finite, so taking limit the as $n \rightarrow \infty$ brings it down to zero. For the second term, notice that $\lim _{n \rightarrow \infty} \frac{n-N_{\epsilon}+1}{n}=1$, so in fact we have

$$
\left|b_{n}-L\right|<\epsilon
$$

whenever $n \geq N_{\epsilon}$. In other words, $b_{n}$ satisfies the definition for $\lim b_{n}=L$, as desired.
Notice that in our proof, we didn't need the sequence $a_{n}$ to be positive, so in fact our intuition regarding the average value of a sequence works in general. The only reason while we were talking about positive sequences in the beginning was because we were taking $n$-th roots. Also, notice Lemma 1 fails if any of the terms is zero (because then the product would be zero).
Example 1. We give a quick application of Theorem 1.

$$
\lim _{n \rightarrow \infty} \sqrt[n]{\frac{n^{n}}{n!}}=e
$$

To see this, consider the sequence $a_{n}=n^{n} / n$ !, and take the limit

$$
\lim _{n \rightarrow \infty} \frac{a_{n+1}}{a_{n}}=\lim _{n \rightarrow \infty}\left(1+\frac{1}{n}\right)^{n}=e
$$

