# THE FIBONACCI NUMBERS 

ADRIAN PĂCURAR

## Contents

1. Introduction ..... 2
2. Primer On Generating Functions ..... 2
3. Deriving Binet's Formula ..... 3
4. The Golden Ratio ..... 5
5. Series Of Reciprocals of Fibonacci Numbers ..... 6

## 1. Introduction

One of the most famous number sequence appearing in mathematics is the Fibonacci sequence, named after Leonardo de Pisa (1170), a famous mathematician during the European Middle Ages. Leonardo was born in the Bonacci family, and Fibonacci is short for Filius Bonacci (son of Bonacci).

He wrote Liber Abaci (1202), where he introduces this famous sequence

$$
0,1,1,2,3,5,8,13,21,34,55, \ldots
$$

as an arithmetic problem about counting pairs of rabbits. The pattern here is that each term is the sum of the previous two. This means we can define it recursively by setting $F_{0}=1$, $F_{1}=1$, and for $n \geq 2$,

$$
F_{n}=F_{n-1}+F_{n-2}
$$

It turns out there is a closed form expression for $F_{n}$. This is the well-known Binet's formula

$$
F_{n}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\sqrt{5}}
$$

which can be easily seen for $n=0$ and $n=1$, but it not obvious at all why the above expression evaluates to an integer, especially when we have $\sqrt{5}$ appear. In order to fully understand this result, we first need to learn about generating functions.

## 2. Primer On Generating Functions

One useful tool for studying sequences of real numbers $\left\{a_{n}\right\}_{n=0}^{\infty}=\left\{a_{0}, a_{1}, a_{2}, \ldots\right\}$ is a generating function. This is nothing than a power series in the variable $x$ whose coefficients are the terms in the sequence:

$$
A(x)=\sum_{n=0}^{\infty} a_{n} x^{n}=a_{0}+a_{1} x+a_{2} x^{2}+a_{3} x^{3}+\ldots
$$

Usually, we are not interested in the question of convergence for these series. We are simply interested in the coefficients it encodes.

Manipulating the power series corresponds to performing certain operations on the sequence of coefficients we are working with. Starting with two sequences $\left\{a_{n}\right\}_{n \geq 0}=\left\{a_{0}, a_{1}, a_{2}, \ldots\right\}$ and $\left\{b_{n}\right\}_{n \geq 0}=\left\{b_{0}, b_{1}, b_{2}, \ldots\right\}$, with associated generating functions $A(x)$ and $B(x)$, respectively, we have the following:
(a) Inserting a zero at the beginning of the sequence:

$$
\left\{0, a_{0}, a_{1}, \ldots\right\} \longleftrightarrow x A(x)
$$

(b) Removing the first element of a sequence:

$$
\left\{a_{1}, a_{2}, a_{3}, \ldots\right\} \longleftrightarrow \frac{A(x)-a_{0}}{x}
$$

(c) Adding two sequences together, i.e. $\left\{a_{n}+c \cdot b_{n}\right\}_{n \geq 0}$ where $c$ is a constant:

$$
\begin{gathered}
\left\{a_{0}+c b_{0}, a_{1}+c b_{1}, a_{2}+c b_{2}, \ldots\right\} \longleftrightarrow A(x)+c B(x) \\
2
\end{gathered}
$$

(d)

$$
\left\{(n+1) a_{n+1}\right\}_{n \geq 0} \longleftrightarrow A(x)+c B(x)
$$

(e)

$$
\left\{0, a_{0}, \frac{a_{1}}{2}, \frac{a_{2}}{3}, \frac{a_{3}}{4} \ldots\right\} \longleftrightarrow \int_{0}^{x} A(t) d t
$$

(f) Convolution of two sequences, i.e. $c_{n}=\sum_{k}=0^{n} a_{k} b_{n-k}$ :

$$
\left\{c_{n}\right\}_{n \geq 0} \longleftrightarrow A(x) B(x)
$$

As an exercise, observe that the sequence of partial sums $c_{n}=\sum_{k=0}^{n} a_{k}$ is given by

$$
\left\{c_{n}\right\}_{n \geq 0} \longleftrightarrow \frac{A(x)}{1-x}
$$

The next section illustrates how the method of generating functions is used to obtain information about certain sequences.

## 3. Deriving Binet's Formula

We begin with the generating function for the Fibonacci numbers $\{0,1,1,2,3,5,8, \ldots\}$ :

$$
F(x)=F_{0}+F_{1} x+F_{2} x^{2}+\cdots=0+x+x^{2}+2 x^{3}+3 x^{4}+5 x^{5}+8 x^{6}+\ldots
$$

We would like to manipulate this in such a way that allows us to exploit the recurrence $F_{n}=F_{n-1}+F_{n-2}$. Observe that we can remove the first term by doing

$$
\frac{F(x)-0}{x}=F_{1}+F_{2} x+F_{3} x^{2}+\ldots
$$

and now we can add the above power series. When we combine the like powers of $x$, we get

$$
F(x)+\frac{F(x)}{x}=\left(F_{0}+F_{1}\right)+\left(F_{1}+F_{2}\right) x+\left(F_{2}+F_{3}\right) x^{2}+\ldots
$$

which by the recurrence is also equal to $F_{2}+F_{3} x+F_{4} x^{2}+\ldots$, but this is nothing other than the original series with the first two terms removed, and divited by $x_{2}$, so we have the functional equation:

$$
F(x)+\frac{F(x)}{x}=\frac{F(x)-x}{x^{2}}
$$

Multiplying through by $x^{2}$, we have

$$
x^{2} F+x F-F=-x \Longleftrightarrow\left(x^{2}+x-1\right) F=-x
$$

and so the closed formula for the Fibonacci generating function is going to be

$$
F(x)=\frac{x}{1-x-x^{2}}
$$

But now notice that the denominator is a parabola with a $y$-intercept equal to 1 , and $\lim _{x \rightarrow \pm \infty} 1-x-x^{2}=-\infty$. Thus it has two real roots $r_{1}$ and $r_{2}$, so it can be factored as

$$
1-x-x^{2}=\left(1-\frac{x}{r_{1}}\right)\left(1-\frac{x}{r_{2}}\right)
$$

We can solve for $1-x-x^{2}=0$ using the quadratic formula, and we obtain

$$
r_{1}=-\frac{1+\sqrt{5}}{2} \quad \text { and } \quad r_{2}=-\frac{1-\sqrt{5}}{2}
$$

These roots look strikingly similar to the irrational terms in Binet's formula, so we are on the right path. However, they are off by a negative sign, but we can fix that. It turns out that these numbers have a nice property, namely $r_{1} r_{2}=-1$, and so we can replace $x / r_{1}$ by $-r_{2} x$ in the factorization, and do the same with the other. Then we can write

$$
1-x-x^{2}=(1-\alpha x)(1-\beta x)
$$

where

$$
\alpha=\frac{1+\sqrt{5}}{2} \quad \text { and } \quad \beta=\frac{1-\sqrt{5}}{2}
$$

Indeed, we can check our conversion since $\alpha \beta=-1$ (the coefficient of $x^{2}$ ) and $-\alpha-\beta=-1$ (the coefficient of $x$ ). Our generating function becomes

$$
F(x)=\frac{x}{(1-\alpha x)(1-\beta x)}
$$

which we rewrite, using the method of partial fractions, as

$$
F(x)=\frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha x}-\frac{1}{1-\beta x}\right)
$$

This allows us to express our generating function as a combination of geometric series:

$$
F(x)=\frac{1}{\sqrt{5}}\left(\sum_{i \geq 0} \alpha^{i} x^{i}-\sum_{i \geq 0} \beta^{i} x^{i}\right)
$$

On the LHS, the coefficient of $x^{n}$ is given by the $n$-th Fibonacci number $F_{n}$ (this is how we defined our function), while on the RHS it is $\frac{1}{\sqrt{5}}\left(\alpha^{i}-\beta^{i}\right)$. But this is exactly the formula we were trying to derive!

$$
F_{n}=\frac{1}{\sqrt{5}}\left(\alpha^{i}-\beta^{i}\right)=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\sqrt{5}}
$$

At this point, we could also talk about where this power series converges. For that, we need $|x \alpha|<1$ and $|x \beta|<1$, but this is equivalent to

$$
|x|<\frac{2}{1+\sqrt{5}} \quad \text { and } \quad|x|<\frac{2}{\sqrt{5}-1}
$$

and since the the first inequality is more restrictive (larger denominator), the series converges for

$$
|x|<\frac{2}{1+\sqrt{5}}
$$

## 4. The Golden Ratio

It turns out that the values $\alpha$ and $\beta$ we obtained from the previous section are very famous. The value of $\alpha$ is known as the Golden Ratio $\phi$, and is equal to

$$
\phi=\frac{1+\sqrt{5}}{2} \approx 1.61803399 \ldots
$$

while the value of $\beta$ is simply $-1 / \phi$ or $1-\phi$. The number $\phi$ appears in many forms in nature, such as proportions in the human body.

One other way $\phi$ arises from the Fibonacci sequence is when we look at ratios of consecutive terms:

| $n$ | $F_{n+1} / F_{n}$ |
| :---: | :--- |
| 1 | $F_{2} / F_{1}=1 / 1=\mathbf{1}$ |
| 2 | $F_{3} / F_{2}=2 / 1=\mathbf{2}$ |
| 3 | $F_{4} / F_{3}=3 / 2=1.5$ |
| 4 | $F_{5} / F_{4}=5 / 3 \approx \mathbf{1 . 6 6 6 6 7}$ |
| 5 | $F_{6} / F_{5}=8 / 5=\mathbf{1 . 6}$ |
| 6 | $F_{7} / F_{6}=13 / 8=1.625$ |
| 7 | $F_{8} / F_{7}=21 / 13 \approx \mathbf{1 . 6 1 5 3 8}$ |
| 8 | $F_{9} / F_{8}=34 / 21 \approx \mathbf{1 . 6 1 9 0 5}$ |
| 9 | $F_{10} / F_{9}=55 / 34 \approx 1.61765$ |
| 10 | $F_{11} / F_{10}=89 / 55 \approx \mathbf{1 . 6 1 8 1 8}$ |
| 11 | $F_{12} / F_{11}=144 / 89 \approx \mathbf{1 . 6 1 8 1 0}$ |

It appears that the ratio stabilizes around 1.618 as $n$ gets larger, which is close to the value of the Golden ratio. But this is easy to prove. Assuming $F_{n+1} / F_{n}$ has a limit $L$, we have

$$
L=\lim _{n \rightarrow \infty} \frac{F_{n+1}}{F_{n}}
$$

and now we simply use the recurrence in the numerator, to get

$$
L=\lim _{n \rightarrow \infty} \frac{F_{n}+F_{n-1}}{F_{n}}
$$

Splitting the fraction and noting that if $F_{n+1} / F_{n} \rightarrow L$, then $F_{n-1} / F_{n} \rightarrow 1 / L$ as $n \rightarrow \infty$, we obtain almost the same quadratic as before,

$$
L=1-\frac{1}{L} \Longleftrightarrow L^{2}=L+1 \Longleftrightarrow L^{2}-L-1=0
$$

off by a factor of $L^{2}$. But the roots are the same, and since our limit $L$ should be positive, we pick the positive root, which is the Golden Ratio:

$$
L=\phi=\frac{1+\sqrt{5}}{2}
$$

## 5. Series Of Reciprocals of Fibonacci Numbers

In calculus, we study the harmonic series, and in general $p$-series of the form

$$
\sum \frac{1}{n^{p}}
$$

which converge whenever $p>1$. In particular, the series of reciprocals of integers, $\sum 1 / n$, is divergent. One natural question to ask as a calculus student is whether the series of reciprocals of Fibonacci numbers converges or diverges:

$$
\sum_{n=1}^{\infty} \frac{1}{F_{n}}
$$

This is actually pretty easy to answer, because we can use Binet's formula and observe that as $n \rightarrow \infty$,

$$
F_{n} \sim \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n}=\frac{\phi^{n}}{\sqrt{5}}
$$

as the other term in the numerator is less than 1 , so it goes to zero. But then we can estimate our series as a geometric series with common ratio $1 / \phi<1$, and so $\sum \frac{1}{F_{n}}$ converges.

A more precise argument is as follows, and it bounds the series above:

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{F_{n}} & =\sum_{n=1}^{\infty} \frac{\sqrt{5}}{\phi^{n}-\left(\frac{-1}{\phi}\right)^{n}} \\
& <\sum_{n=1}^{\infty} \frac{\sqrt{5}}{\phi^{n}-\left(\frac{1}{\phi}\right)^{n}} \\
& =\sum_{n=1}^{\infty} \frac{\sqrt{5}}{\phi^{n}\left[1-\left(\frac{1}{\phi^{2}}\right)^{n}\right]} \\
& <\sum_{n=1}^{\infty} \frac{\sqrt{5}}{\phi^{n}}
\end{aligned}
$$

and the last series is clearly convergent, bounding $\sum \frac{1}{F_{n}}$ above.

There are a lot more things to say about the Fibonacci numbers. One of the most complete resources for the interested reader is Fibonacci and Lucas Numbers with Applications by Thomas Koshy. And of course, there's always Wikipedia.

