Graph Stirling Numbers

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Classical Stirling Numbers

Definition: For $n, k \geq 1$, the *Stirling numbers of the first kind*, $s(n,k) = \left[{n \atop k}\right]$, count the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k cycles.

For $n, k \geq 1$, the Stirling numbers of the first kind satisfy

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix},$$

with initial conditions $\left[{0 \atop 0}\right] = 1$ and $\left[{* \atop 0}\right] = \left[{0 \atop *}\right] = 0$.

Classical Stirling Numbers

Definition: For $n, k \geq 1$, the *Stirling numbers of the second kind*, $S(n,k) = \left\{ {n \atop k} \right\}$, count the number of partitions of $\{1, 2, \ldots, n\}$ into k nonempty parts.

For $n, k \geq 1$, the Stirling numbers of the second kind satisfy

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

with initial conditions $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {* \atop 0} \right\} = \left\{ {0 \atop *} \right\} = 0$.

Stirling Matrices

The 5×5 Stirling matrices of the first and second kind are

$$s = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 6 & 11 & 6 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 7 & 6 & 1 \end{pmatrix}$$

Theorem: For any $n > 0$, the $n \times n$ Stirling matrices satisfy

$$\left(\left\{ {m \atop k} \right\} \right)^{-1}_{0 \le m,k \le n} = \left((-1)^{m+k} \left[{m \atop k} \right] \right)_{0 \le m,k \le n}$$

- Any $(n-1) \times (n-1)$ minor is nonnegative
- In fact, every minor is nonnegative!

Totally Nonnegative Matrices

Definition: A matrix is said to be *totally nonnegative* (TN) if each of its minors is nonnegative.

Example: One can easily check that the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix}$$

is totally nonnegative by verifying that every minor is nonnegative.

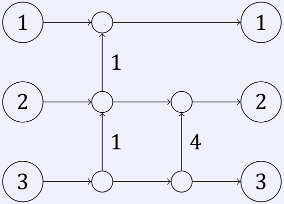
Theorem: For any $n \geq 1$, the $n \times n$ Stirling matrices are TN.
Proof: Uses planar networks.

Planar Networks

Definition: A *planar network of order* n is an acyclic, planar, weighted, directed graph with $2n$ designated boundary vertices (n sources and n sinks), $s_1, \ldots s_n$ and $t_1, \ldots, t_n$.

For any planar network G, we can construct the *path matrix* $(w_{i,j})$, where $w_{i,j}$ counts the weighted number of paths from s_i to t_j.

Example: A planar network and its associated path matrix:



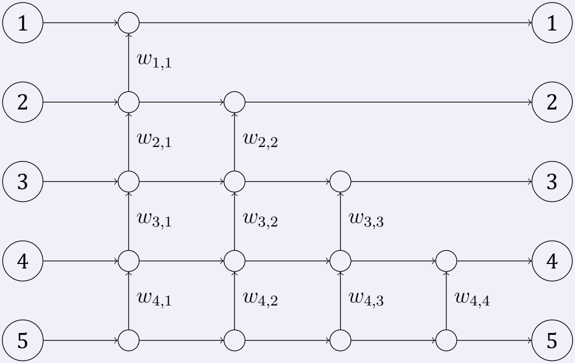
$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix}$$

Note: We also found this matrix to be TN.

Planar Networks And Total Nonnegativity

Theorem: (Lindstrőm's Lemma) The path matrix of any planar network with nonnegative weights is TN.

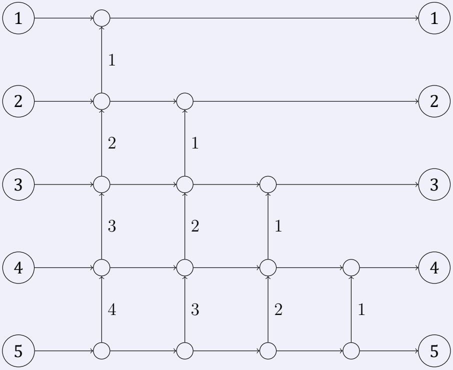
Note: In general, different networks can give rise to the same path matrix, but if our matrix is invertible and lower triangular, we can use:



Stirling Total Nonnegativity

Theorem: For any $n \geq 0$, the $n \times n$ Stirling matrices $\left(\left[{m \atop k}\right]\right)_{1 \leq m,k \leq n}$ and $\left(\left\{{m \atop k}\right\}\right)_{1 \leq m,k \leq n}$ are TN.

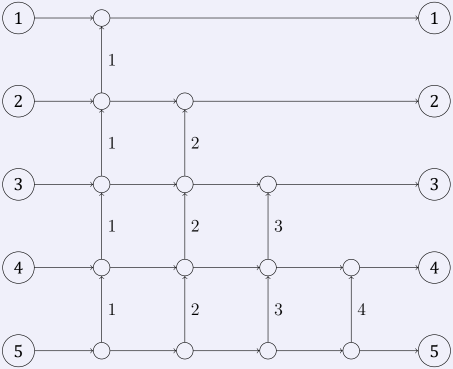
Proof: The planar network for the Stirling numbers of the first kind is:



Stirling Total Nonnegativity

Theorem: For any $n \geq 0$, the $n \times n$ Stirling matrices $\left(\begin{bmatrix} m \\ k \end{bmatrix}\right)_{1 \leq m,k \leq n}$ and $\left(\begin{Bmatrix} m \\ k \end{Bmatrix}\right)_{1 \leq m,k \leq n}$ are TN.

Proof: The planar network for the Stirling numbers of the second kind is:



Graph Stirling Numbers

Definition: For any graph G on n vertices $\{v_1, \ldots, v_n\}$, define the *graph Stirling numbers of the second kind* $\left\{ {G \atop k} \right\}$ to be the number of ways of partitioning the vertices of G into k nonempty independent sets.

Typically we consider G_m to be the induced subgraph of G on vertices $\{v_1, v_2, \ldots, v_m\}$, and we look at $\left\{ {G_m \atop k} \right\}$.

Example: For G an independent set, this gives just the classical Stirling numbers (the empty edge structure imposes no restriction on partitioning), so $\left\{ {G_m \atop k} \right\} = \left\{ {m \atop k} \right\}$.

Example: For G the complete graph, we have

$$\left\{ {K_m \atop k} \right\} = \delta_{m,k} = \begin{cases} 0 & m \neq k \\ 1 & m = k \end{cases}$$

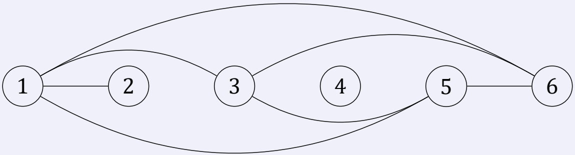
Chordal Graphs

Definition: A *chordal graph* is a graph in which every cycle of length at least 4 has a chord.

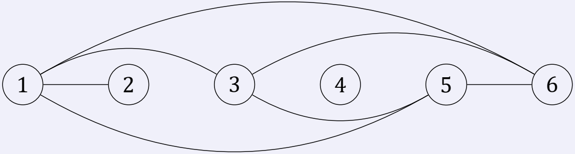
Definition: A *perfect elimination order* of a graph is a labeling of the vertices such that for any vertex v_i, its preceeding neighbours in the labeling form a clique.

Theorem: (Characterization) A *chordal graph* is a graph that admits a perfect elimination order.

Example:



Chordal Graphs



$$S_G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & 6 & 6 & 1 & 0 \\ 0 & 0 & 0 & 12 & 8 & 1 \end{pmatrix}, \quad S_G^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -4 & 1 & 0 & 0 \\ 0 & -6 & 18 & -6 & 1 & 0 \\ 0 & 24 & -96 & 36 & -8 & 1 \end{pmatrix}$$

Total Nonnegativity Of Graph Stirling

Theorem: If G is chordal on n vertices listed in a perfect elimination order, then the matrix $\left(\left\{ {G_m \atop k} \right\}\right)_{0 \le m,k \le n}$ is TN. This generalizes the classical result.

Proof: Also uses planar networks.

- Construct a planar network with path matrix $\left(\left\{ {G_m \atop k} \right\}\right)$
- Some of the weights are negative
- Manipulate the weights to turn them all positive without changing the path matrix

Thank you!