# NUMBER THEORETIC FUNCTIONS AND THE DIRICHLET GENERATING FUNCTION

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## 1. Introduction

Questions about the divisors $d$ of an integer $n$ are at the heart of number theory. We can define many functions $f : \mathbb{N} \to \mathbb{R}$ that give us information about the divisors of $n$.

**Definition.** The *divisor function* $\tau : \mathbb{N} \to \mathbb{N}$ counts the number of divisors of $n$. We have

$$\tau(n) = \sum_{d|n} 1$$

where the sum is taken over all positive divisors $d$ of $n$.

**Example.** $\tau(8) = 4$, since 8 has 4 divisors $\{1, 2, 4, 8\}$

**Example.** $\tau(12) = 6$, since 12 has 6 divisors $\{1, 2, 3, 4, 6, 12\}$.

**Definition.** The *divisor sum function* $\sigma : \mathbb{N} \to \mathbb{N}$ is the sum of the divisors of $n$, given by

$$\sigma(n) = \sum_{d|n} d$$

where the sum is again taken over all positive divisors $d$ of $n$.

**Example.**

- $\sigma(2) = 1 + 2 = 3$
- $\sigma(3) = 1 + 3 = 4$
- $\sigma(8) = 1 + 2 + 4 + 8 = 15$

The following table computes $\tau$ and $\sigma$ for $1, 2, 3, \ldots, 12$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau(n)$ | 1 | 2 | 2 | 3 | 2 | 4 | 2 | 4 | 3 | 4 | 2 | 6 |
| $\sigma(n)$ | 1 | 3 | 4 | 7 | 6 | 12 | 8 | 15 | 13 | 18 | 12 | 28 |

We immediately see that if $p$ is prime, $\tau(p) = 2$ and $\sigma(p) = 1+p$, as the only positive divisors of $p$ are 1 and $p$. We can generalize this for powers of $p$.

**Lemma 1.** For $p$ prime and $k > 0$, $\tau(p^k) = k + 1$ and $\sigma(p^k) = \dfrac{p^{k+1} - 1}{p - 1}$.

*Proof.* Let $D(p^k) = \left\{1, p, p^2, p^3, \ldots, p^k\right\}$ be the set of all positive divisors of $p^k$. Then

$$\tau(p^k) = |D(p^k)| = k + 1$$

by definition. To compute $\sigma(p^k)$, we use geometric sums:

$$\sigma(p^k) = \sum_{r \in D(p^k)} r = 1 + p + p^2 + \cdots + p^k = \frac{p^{k+1} - 1}{p - 1}$$

which concludes our proof. $\square$

Notice how $\tau(6) = \tau(2)\tau(3)$, $\tau(10) = \tau(2)\tau(5)$, but this pattern doesn't always hold: $\tau(12) = 6 \neq \tau(2)\tau(6) = 2 \cdot 4 = 8$. It turns out that if $m, n$ are relatively prime, $\tau(mn) = \tau(m)\tau(n)$. We have the following theorem:

**Theorem 1.** For distinct primes $p_1, p_2, \dots p_r$, and positive integers $k_1, k_2, \dots, k_r$, we have

$$\tau(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) = \tau(p_1^{k_1})\tau(p_2^{k_2}) \cdots \tau(p_r^{k_r})$$

*Proof.* We use a simple counting argument. Let $N = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, and consider the sets $D_i = \left\{1, p_i, p_i^2, \dots, p_i^{k^i}\right\}$ for $i = 1, 2, \dots, r$. Any divisor of of $N$ will be a product of up to $k_i$ elements from each $D_i$, and so the set $D(N)$ of all positive divisors of $N$ has size

$$\tau(N) = |D(N)| = |D_1| \cdot |D_2| \cdots |D_r| = \prod_{i=1}^{r}(k_i + 1) = \tau(p_1^{k_1})\tau(p_2^{k_2}) \cdots \tau(p_r^{k_r})$$

where the last equality follows from Lemma 1. □

A closer look at our table suggests that, similarly to $\tau$, in general we do not have $\sigma(mn) = \sigma(m)\sigma(n)$, but it turns out that this result does hold for $m$ and $n$ are relatively prime.

**Lemma 2.** Let $m$ and $n$ be relatively prime. Then $\sigma(mn) = \sigma(m)\sigma(n)$.

*Proof.* Let $D(m) = \{1, m_1, m_2, \dots, m\}$ be the set of all positive divisors of $m$, and $D(n) = \{1, n_1, n_2, \dots, n\}$ be the set of all positive divisors of $n$.

Observe that any divisor of $mn$ is of the form $m_i n_j$, where $m_i \in D(m)$ and $n_j \in D(n)$. The sum of the divisors of $m$ and $n$, respectively, are

$$\sigma(m) = 1 + m_1 + m_2 + \cdots + m \quad \text{and} \quad \sigma(n) = 1 + n_1 + n_2 + \cdots + n$$

It is easy to see that multiplying these two expression gives a sum of all possible combinations $m_i n_j$, and so

$$\sigma(mn) = \sum_{i,j} m_i n_j = (1 + m_1 + m_2 + \cdots + m)(1 + n_1 + n_2 + \cdots + n) = \sigma(m)\sigma(n)$$

as desired. □

**Theorem 2.** For distinct primes $p_1, p_2, \dots p_r$, and positive integers $k_1, k_2, \dots, k_r$, we have

$$\sigma(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) = \sigma(p_1^{k_1})\sigma(p_2^{k_2}) \cdots \sigma(p_r^{k_r}) = \prod_{i=1}^{r} \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

*Proof.* Follows immediately from lemmas 1 and 2. □

## 2. Primer On Generating Functions

In the introduction, we saw that the sequences

$$\{\tau(n)\}_{n=1}^{\infty} = \{1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, 6, \dots\}$$

and

$$\{\sigma(n)\}_{n=1}^{\infty} = \{1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, \dots\}$$

are quite irregular, even though we were able to obtain some formulas for them. One useful tool for studying sequences of real numbers $\{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, \dots\}$ is a generating function.

**Definition.** The *ordinary generating function* (OGF) of the sequence $\{a_n\}_{n=0}^{\infty}$ is a formal power series in the variable $x$ whose coefficients are the terms in the sequence:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

**Example.** The OGF of the sequence $\{1, 1, 1 \dots\}$ is the function

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

Usually, we are not interested in the question of convergence for these series. We are simply interested in the coefficients it encodes. Manipulating the power series corresponds to performing certain operations on the sequence of coefficients we are working with.

**Example.** Starting with two sequences $\{a_n\}_{n\geq 0} = \{a_0, a_1, a_2, \dots\}$ and $\{b_n\}_{n\geq 0} = \{b_0, b_1, b_2, \dots\}$, with corresponding OGFs $A(x)$ and $B(x)$, respectively, we have the following:

(a) Removing the first element of a sequence:

$$\{a_1, a_2, a_3, \dots\} \longleftrightarrow \frac{A(x) - a_0}{x}$$

(b) Adding two sequences together, i.e. $\{a_n + c \cdot b_n\}_{n\geq 0}$ where $c$ is a constant:

$$\{a_0 + cb_0, a_1 + cb_1, a_2 + cb_2, \dots\} \longleftrightarrow A(x) + cB(x)$$

(c)

$$\{(n+1)a_{n+1}\}_{n\geq 0} \longleftrightarrow A'(x)$$

(d)

$$\left\{0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \frac{a_3}{4} \dots\right\} \longleftrightarrow \int_0^x A(t)dt$$

**Example.** The *convolution* of two sequences, i.e. $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, has OGF

$$\{c_n\}_{n\geq 0} \longleftrightarrow A(x)B(x)$$

**Example.** Given a sequence $a_n$, observe that the sequence of partial sums $c_n = \sum_{k=0}^{n} a_k$ is given by

$$\{c_n\}_{n\geq 0} \longleftrightarrow \frac{A(x)}{1-x}$$

**Example.** It is widely known that the $n$-th harmonic number

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

is never an integer for $n \geq 2$. What is the OGF of the sequence $\{H_n\}$? By the previous example, it suffices to find the OGF of $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$ and divide it by $1-x$ to get the sequence of partial sums. From calculus, we know that

$$\sum_{n\geq 1} \frac{x^n}{x} = -\log(1-x)$$

and so the OGF must be

$$\sum_{n\geq 1} H_n x^n = \frac{1}{1-x}\log\left(\frac{1}{1-x}\right)$$

There are other types of generating functions, some more useful than others depending on the properties of the sequence we are dealing with.

**Definition.** The *exponential generating function* (EGF) of the sequence $\{a_n\}_{n=0}^{\infty}$ is

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

**Example.** The EGF of the sequence $\{1, 1, 1, \ldots\}$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

**Example.** The EGF of the factorial sequence $\{1, 1, 2, 6, 24, \ldots\}$ is $\sum_{n=0}^{\infty} n! \frac{x^n}{n!} = 1/(1-x)$, which is the same as the OGF of $\{1, 1, 1, \ldots\}$.

**Example.** Starting with a sequence $\{a_n\}_{n\geq 0} = \{a_0, a_1, a_2, \ldots\}$ with EGF $f(x)$, removing the first element (i.e. looking at the sequence $\{a_1, a_2, \ldots\}$) is equivalent to computing $f'(x)$, since

$$f'(x) = \sum_{n=1}^{\infty} a_n \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} a_n \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!}$$

**Example.** The Fibonacci numbers satisfy the recurrence $F_{n+2} = F_{n+1} + F_n$, with initial conditions $F_0 = 0$ and $F_1 = 1$, and by the previous example the EGF of $F_n$ must satisfy the differential equation $f''(x) = f'(x) + f(x)$. I will skip the details here, but solving this differential equation gives the EGF of the Fibonacci sequence

$$f(x) = \frac{1}{\sqrt{5}}\left(e^{(r_+)x} + e^{(r_-)x}\right), \qquad r_{\pm} = \frac{1 \pm \sqrt{5}}{2}$$

This gives a much easier method of solving for Binet's formula, which could be obtained using the OGF, but it involves partial fraction decompositions.

**Example.** Given sequence $\{a_n\}$ and $\{b_n\}$ with EGF $A(x)$ and $B(x)$, what is the sequence whose EGF is the product $AB$? (In the OGF case, we saw this was a convolution.)

$$AB = \sum_{i\geq 0} a_i \frac{x^i}{i!} \sum_{j\geq 0} b_j \frac{x^j}{j!} = \sum_{i,j\geq 0} a_i b_j \frac{x^{i+j}}{i!j!} = \sum_{n\geq 0} x^n \left(\sum_{i+j=n} \frac{a_i b_j}{i!j!}\right) = \sum_{n\geq 0} \frac{x^n}{n!} \left(\sum_{i+j=n} \frac{a_i b_j n!}{i!j!}\right)$$

and so the coefficient of $x^n/n!$ in the product $AB$ is

$$c_n = \sum_k \binom{n}{k} a_k b_{n-k}$$

Hence the resulting sequence is another type of convolution of of $a_n$ and $b_n$.

## 3. The Dirichlet Generating Function

While the OGF and EGF are useful in studying many sequences of numbers, they do not have the desired properties to help us in studying the sequences that arise from the number-theoretic functions $\tau(n)$ and $\sigma(n)$.

**Definition.** The *Dirichlet generating function* (DGF) of the sequence $\{a_n\}_{n=1}^{\infty}$ is

$$A(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = a_1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \frac{a_4}{4^s} + \dots$$

**Example.** The DGF of the sequence $\{1, 1, 1, \dots\}$ is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

which is the *Riemann zeta function*.

Suppose the DGF of sequences $a_n$ and $b_n$ are given by $A(s)$ and $B(s)$, respectively. As we did for the other types of generating functions, we will explore what sequence does the product $AB$ describe. Notice that

$$\begin{aligned} AB &= \big(a_1 + a_2 2^{-s} + a_3 3^{-s} + a_4 4^{-s} + \dots\big)\big(b_1 + b_2 2^{-s} + b_3 3^{-s} + b_4 4^{-s} + \dots\big) \\ &= (a_1 b_1)1^{-s} + (a_1 b_2 + a_2 b_1)2^{-s} + (a_1 b_3 + a_3 b_1)3^{-s} \\ &\quad + (a_1 b_4 + a_2 b_2 + a_4 b_1)4^{-s} + \dots \end{aligned}$$

In general, the coefficent of $n^{-s}$ in the product $AB$ is a sum over all products of the form $a_i b_j$ where the product of $i$ and $j$ is $n$:

$$\sum_{ij=n} a_i b_j$$

Another way to express this is by

$$\sum_{d|n} a_d b_{n/d}$$

which is another type of convolution of sequences $a_n$ and $b_n$, indexed over all positive divisors of $n$. This makes the DGF a promising object for studying the sequences $\tau(n)$ and $\sigma(n)$, which are related to the divisors of $n$.

One natural question that arises from the previous example is the following: suppose $a_n$ has DGF $f(s)$. What sequence does the DGF $f^k(s)$ (the product of $k$ copies of $f$) describe?

$$f^k(s) = \left(\sum_{n\geq 1} a_n n^{-s}\right)^k = \sum_{n_1,\dots,n_k\geq 1} a_{n_1}\cdots a_{n_k}(n_1\dots n_k)^{-s}$$

and if we collect coefficients of $n^{-s}$, this becomes

$$f^k(s) = \sum_{n\geq 1}\left(\sum_{n_1\dots n_k=n} a_{n_1}a_{n_2}\cdots a_{n_k}\right)n^{-s}$$

If we apply this to the sequence $(1, 1, 1, \ldots)$ with DGF $\zeta(s)$, we find that $\zeta^k(s)$ generates a sequence with a nice combinatorial description: the number of ordered factorizations of $n$ into $k$ factors.

In particular, for $k = 2$, $\zeta^2(s)$ generates the sequence

$$\left\{ \sum_{d|n} 1 \cdot 1 \right\}_{n=1}^{\infty}$$

which is the sequence $\tau(n)$ of the divisor function.

## 4. Multiplicative Number-Theoretic Functions

**Definition.** A number-number theoretic function is a function $f : \mathbb{N} \to \mathbb{R}$. Such a function is *multiplicative* whenever $f(mn) = f(m)f(n)$ for $m$ and $n$ relatively prime.

**Example.** We saw that $\tau$ and $\sigma$ are multiplicative. The identity function $f(n) = 1$ is another example of a multiplicative function, and we saw that its DGF is The Riemann zeta function $\zeta(s)$.

One known property of $\zeta(s)$ is that it can be expressed as a product over all primes $p$:

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_p \left( \frac{1}{1 - p^{-s}} \right)$$

and each term can be thought of as the closed form of a geometric series, so we may write

$$\sum_{n \geq 1} n^{-s} = \prod_p \left( 1 + p^{-s} + p^{-2s} + p^{-3s} + \ldots \right)$$

Is it possible to generalize this result to other sequences $f(n)$? The following theorem answers this question.

**Theorem 3.** Let $f$ be a multiplicative number-theoretic function. Then the DGF of the sequence $f(n)$ satisfies the formal identity

$$\sum_{n \geq 1} \frac{f(n)}{n^s} = \prod_p \left( 1 + f(p)p^{-s} + f(p^2)p^{-2s} + f(p^3)p^{-3s} + \ldots \right)$$

The proof of this theorem is rather complicated, and it appears in detail in Wilf. Instead of presenting the proof, we apply the result in order to obtain the DGF of the sequence $\sigma(n)$.

**Theorem 4.** The DGF of the $\sigma(n)$ sequence is $\zeta(s)\zeta(s-1)$.

*Proof.* Recall that for any prime $p$, $\sigma(p^k) = 1 + p + p^2 + \cdots + p^k$. We are interested in finding

$$F(s) = \sum_{n \geq 1} \frac{\sigma(n)}{n^s}$$

and by Theorem 3 this should equal to

$$\sum_{n \geq 1} \frac{\sigma(n)}{n^s} = \prod_p \left( 1 + \sigma(p)p^{-s} + \sigma(p^2)p^{-2s} + \sigma(p^3)p^{-3s} + \ldots \right)$$

which amounts to understanding every term of the form

$$1 + \sigma(p)p^{-s} + \sigma(p^2)p^{-2s} + \sigma(p^3)p^{-3s} + \dots$$

Using the geometric sum expansion for $\sigma(p^k)$, this term becomes the infinite sum

$$\begin{aligned} &(1) \\ &+ (1+p)p^{-s} \\ &+ (1+p+p^2)p^{-2s} \\ &+ (1+p+p^2+p^3)p^{-3s} \\ &+ (1+p+p^2+p^3+p^4)p^{-4s} \\ &\vdots \end{aligned}$$

Instead of summing this infinite triangular array of terms horizontally, we distribute $p^{-ks}$ and we sum the terms vertically. Notice that every vertical row corresponds to a geometric series with common ratio $p^{-s}$, so which gives

$$\frac{1}{1-p^{-s}} + \frac{p^{-s}p^1}{1-p^{-s}} + \frac{p^{-2s}p^2}{1-p^{-s}} + \frac{p^{-3s}p^3}{1-p^{-s}} + \frac{p^{-4s}p^4}{1-p^{-s}} + \dots$$

Now, the numerators also form a geometric series with common ration $p^{-s+1} = p^{-(s-1)}$, so every term in the product is equal to

$$\frac{1}{1-p^{-s}} \cdot \frac{1}{1-p^{-(s-1)}}$$

Finally, our DGF can be expressed as

$$F(s) = \prod_p \left( \frac{1}{1-p^{-s}} \cdot \frac{1}{1-p^{-(s-1)}} \right) = \prod_p \left( \frac{1}{1-p^{-s}} \right) \cdot \prod_p \left( \frac{1}{1-p^{-(s-1)}} \right) = \zeta(s)\zeta(s-1)$$

which concludes our proof. □

## 5. The Möbius Function and Euler's Totient Function

The proof of Theorem 4 can be extended for other multiplicative functions. For prime powers, the Möbius function $\mu(n)$ is equal to

$$\mu(p^k) = \begin{cases} +1 & k = 0 \\ -1 & k = 1 \\ 0 & k \geq 0 \end{cases}$$

It is not too difficult to show that the DGF of the sequence $\mu(n)$ is equal to

$$\sum_{n \geq 1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

Furthermore, a very similar but much easier calculation to Theorem 4 shows that the DGF of Euler's totient function $\phi(n)$, which satisfies $\phi(p^k) = p^k - p^{k-1}$, is given by

$$\sum_{n\geq 1} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$$

We can think of this as a convolution of the sequences whose DGF are $\zeta(s-1)$ and $1/\zeta(s)$. What sequence corresponds to $\zeta(s-1)$? It is easy to see that this is the DGF of $a_n = n$:

$$\sum_{n\geq 1} \frac{n}{n^s} = \sum_{n\geq 1} \frac{1}{n^{s-1}} = \zeta(s-1)$$

Recall that $\mu(n)$ had the DGF $1/\zeta(s)$. Then $\phi(n)$ is a Dirichlet convolution of $a_n = n$ and $\mu(n)$, which immediately gives the famous result:

$$\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}$$

What about the Dirichlet convolution of $\phi(n)$ with the constant sequence $a_n = 1$? This is the sequence whose DGF is the product

$$\frac{\zeta(s-1)}{\zeta(s)} \cdot \zeta(s) = \zeta(s-1)$$

which is the DGF of the sequence $\{n\}$. But this now proves another famous result

$$\sum_{d|n} \phi(d) = n$$

So we see that this intersection of number theory, combinatorics, and analysis can be used to prove some very powerful results in an elegant fashion, which would otherwise require tedious divisibility arguments.

One last very quick application is a proof of the Möbius Inversion Formula.

**Theorem 5.** (Möbius Inversion) Suppose $f$ and $g$ are number-theoretic functions satisfying

$$g(n) = \sum_{d|n} f(d)$$

Then

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$$

*Proof.* Suppose $F(s)$ and $G(s)$ are the DGF of sequences $f(n)$ and $g(n)$ respectively. If $f$ and $g$ are related by $g(n) = \sum_{d|n} f(d)$, then $g$ is a Dirichlet convolution of $f$ with $a_n = 1$, so $G(s) = F(s) \cdot \zeta s$. But then

$$F(s) = \frac{G(s)}{\zeta(s)}$$

so $f$ is a Dirichlet convolution of $g(n)$ with $\mu(n)$ (whose DGF is $1/\zeta$). □

**Theorem 6.** For any $n \geq 1$,

$$\sum_{d|n} \mu(d)\tau\left(\frac{n}{d}\right) = 1$$

*Proof.* the DGF of $\mu(n)$ is $1/\zeta(s)$, while the DGF of $\tau(n)$ is $\zeta^2(s)$. Their product is $\zeta(s)$, which is the DGF of the constant sequence $a_n = 1$. $\square$

## 6. References


1. Tom Apostol, *Introduction to Analytic Number Theory.*
2. Jeffrey Stopple, *A Primer of Analytic Number Theory.*
3. Herbert S. Wilf, *generatingfunctionology.*