Probabilistic Proofs

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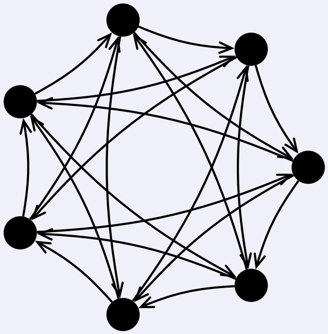
Tournaments

Definition: A *tournament* is a complete directed graph $T = (V, E)$ on a set of n vertices V, with a set of directed edges E so that for every distinct $u, v \in V$, either $(u, v) \in E$ or $(v, u) \in E$.

Example: The following is a tournament with 7 vertices.



Tournaments

One may think of the vertices V as being the players. The edge set *encodes the outcome of every game*: the presence of (u, v) indicates that u beats v, while (v, u) indicates the opposite outcome.

In a tournament with n players, we have

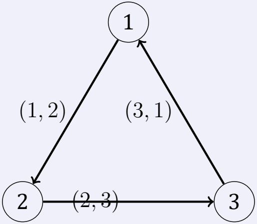
- $\binom{n}{2}$ edges (total games played)
- $2^{\binom{n}{2}}$ possible tournaments

Q: Is there a dominating player? Answer: not always.

Tournaments And Dominating Players

Definition: We say a tournament *has property* D_k if for every subset $K \subseteq V$ of k players, there is a dominating player (in $V \setminus K$) that beats all.

Example: The following tournament has property D_1.



Q: For every finite k, is there a tournament (on more than k vertices) with property D_k? How many vertices do we need?

Tournaments And Dominating Players

Theorem (Erdős, 1963): If $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there exists a tournament on n vertices with property D_k.

Proof: Consider a random tournament on $V = \{1, 2, \ldots, n\}$.

- Decide the outcome of every game by flipping a fair coin
- For every subset $K \subseteq V$ of size k, define the event

$$A_K : \text{the event that no player dominates } K$$

- For an outside player $u \in V \setminus K$, the probability that u beats all players in K is $(1/2)^k = 2^{-k}$, and so

$$P(u \textit{ does NOT dominate } K) = 1 - 2^{-k}$$

- What is $P(A_K)$? There are $n-k$ players outside K, so

$$P(A_K) = \left(1 - 2^{-k}\right)^{n-k}$$

Tournaments And Dominating Players

Theorem (Erdős, 1963): If $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there exists a tournament on n vertices with property D_k.

Proof: We saw that for every subset K, the event A_K that no player dominates K occurs with probability

$$P(A_K) = \left(1 - 2^{-k}\right)^{n-k}$$

Q: What is the probability that we don't have property D_k? The probability that one or more of these subsets K do not have a dominating vertex is

$$P\left(\bigcup_{\substack{K \subseteq V \\ |K|=k}} A_K\right) \leq \sum_{\substack{K \subseteq V \\ |K|=k}} P(A_K) = \binom{n}{k}\left(1 - 2^{-k}\right)^{n-k}$$

Tournaments And Dominating Players

Theorem (Erdős, 1963): If $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, then there exists a tournament on n vertices with property D_k.

Proof: We saw that a randomly constructed tournament will lack property D_k with probability

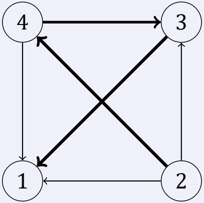
$$P(\text{no } D_k) = \binom{n}{k}\left(1-2^{-k}\right)^{n-k}$$

- If this is smaller than 1, this guarantees the existence of some tournament on n vertices with property D_k
- $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$ is polynomial in n of degree k
- $\left(1-2^{-k}\right)^{n-k}$ is exponential in n with base < 1
- The desired condition occurs for sufficiently large n

Hamiltonian Paths In Tournaments

Definition: A *Hamiltonian path* in a tournament is a permutation $\left\{v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)}\right\}$ of the vertices such that $v_{\pi(i)}$ beats $v_{\pi(i+1)}$ for every i.

Example:



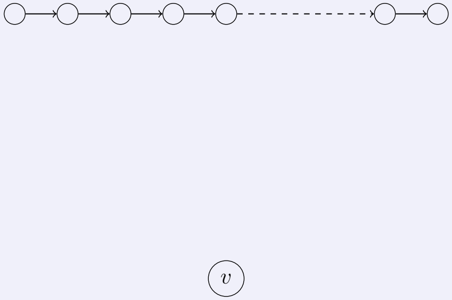
The tournament shown contains one H-path:

$$2 \to 4 \to 3 \to 1$$

Hamiltonian Paths In Tournaments

Theorem (Rédei, 1934): Every tournament T contains a Hamiltonian path.

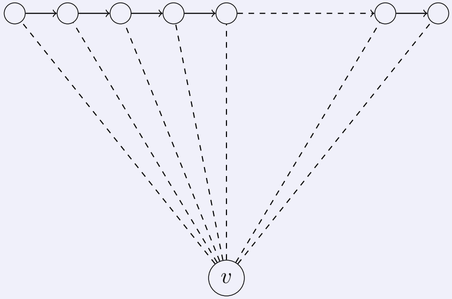
Proof: Consider a path that misses a vertex v:



Hamiltonian Paths In Tournaments

Theorem (Rédei, 1934): Every T contains a H-path.

Proof: Any path that misses a vertex v can be modified to include that vertex:



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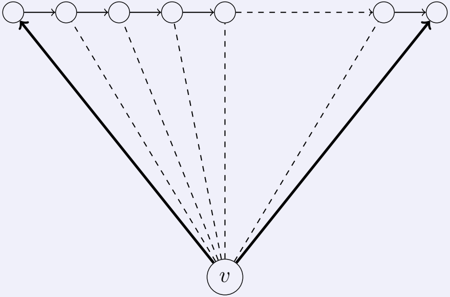

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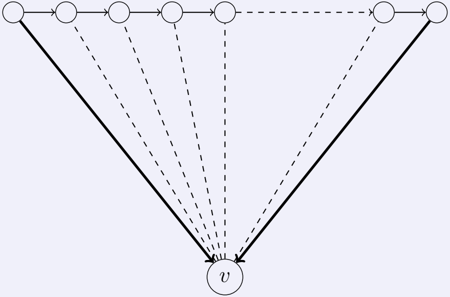

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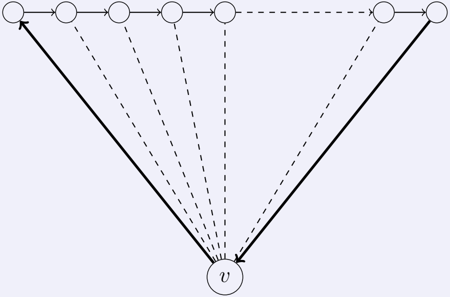

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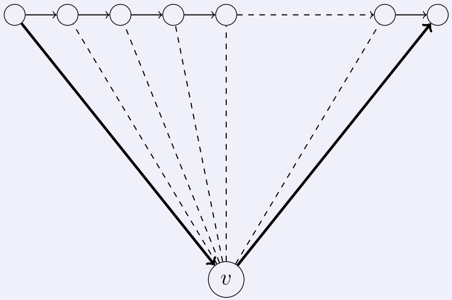

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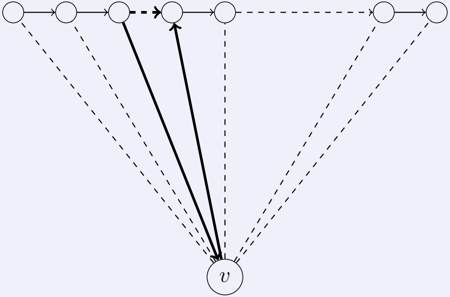

Hamiltonian Paths In Tournaments

Q: Are there tournaments with a lot of H-paths?

Theorem (Szele, 1943): There exist a tournament with n players and $n!/2^{n-1}$ H-paths.

Proof: Construct a random tournament T.

- Define a random variable $X =$ the # of H-paths in T
- For every permutation $\pi \in S_n$ of the vertices, define the indicator random variable

$$\chi_\pi = \begin{cases} 1 & \text{if } \pi \text{ forms a H-path} \\ 0 & \text{otherwise} \end{cases}$$

- Observe that X can be written as

$$X = \sum_\pi \chi_\pi$$

Hamiltonian Paths In Tournaments

Theorem (Szele, 1943): There exist a tournament with n players and $n!/2^{n-1}$ H-paths.

Proof: What is $E(X)$?

$$\begin{aligned} E(X) &= E\left(\sum_{\pi} \chi_{\pi}\right) \\ &= \sum_{\pi} E(\chi_{\pi}) \text{ (linearity of expectation)} \\ &= \sum_{\pi} \frac{1}{2^{n-1}} \text{ (there are } n-1 \text{ possible edges)} \\ &= \frac{n!}{2^{n-1}} \end{aligned}$$

At least one tournament will achieve or surpass this bound.

Sum-Free Sets

Definition: A subset S of an additive group is *sum-free* whenever $x + y \notin S$ for all $x, y \in S$.

Example: The odd integers are sum-free.

Theorem (Erdős, 1965): Let $A \subseteq \mathbb{N}$ be a set of N nonzero integers. Then there is a sum-free $S \subseteq A$ with $|S| > N/3$.

Proof:

- Start with $A \subseteq \mathbb{N}$ with N elements
- Pick a prime $p = 3k + 2$ with $p > 2 \cdot \max_{a \in A} \{|a|\}$
- Consider $B = \{k + 1, k + 2, \ldots, k + (k + 1)\}$, which is a sum-free subset of $\mathbb{Z}/p\mathbb{Z}$
- Pick $t \in (\mathbb{Z}/p\mathbb{Z})^\times$ uniformly at random, and let

$$A_t = \{a \in A : at \in B \pmod{p}\}$$

Sum-Free Sets

Theorem (Erdös, 1965): Let $A \subseteq \mathbb{N}$ be a set of N nonzero integers. Then there is a sum-free $S \subseteq A$ with $|S| > N/3$.

Proof: So far we have:

- $B = \{k+1, k+2, \ldots, k+(k+1)\}$ sum free
- $A_t = \{a \in A : at \in B \pmod{p}\}$ is also sum-free (since its residues modulo p belong to B)
- Goal: show that A_t is large for some t
- For any fixed $a \in A$, and look at $a \cdot t$:

$$P[at \in B \pmod{p}] = \frac{|B|}{p-1} = \frac{k+1}{3k+1} > \frac{1}{3}$$

(recall $t \in (\mathbb{Z}/p\mathbb{Z})^{\times}$, $0 \notin A$, and t was uniformly chosen)

Sum-Free Sets

Theorem (Erdős, 1965): Let $A \subseteq \mathbb{N}$ be a set of N nonzero integers. Then there is a sum-free $S \subseteq A$ with $|S| > N/3$.

Proof:

- We saw that for any fixed $a \in A$ and random t,

$$P[at \in B \text{ (mod } p)] > 1/3$$

- What is the expectation $E(|A_t|)$?

$$\begin{aligned} E(|A_t|) &= \sum_{a \in A} P(a \in A_t) \\ &= \sum_{a \in A} P[at \in B \text{ (mod } p)] > \frac{|A|}{3} \end{aligned}$$

Thank You!